DE GRUYTER
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DOI 10.1515/tmj-2017-0048

A fixed point approach to Ulam-Hyers stability of duodecic functional equation in quasi-β-normed spaces

J. M. Rassias[1], K. Ravi[2] and B. V. Senthil Kumar[3]

1Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens, Attikis 15342, Greece
[2]Department of Mathematics, Sacred Heart College, Tirupattur-635 601, TamilNadu, India
[3] Department of Mathematics, C. Abdul Hakeem College of Engineering and Technology, Melvisharam-632 509, Tamil Nadu, India
E-mail: jrassias@primedu.uoa.gr[1], shckravi@yahoo.co.in[2], bvskumarmaths@gmail.com[3]

Abstract

In this study, we achieve the general solution and investigate Ulam-Hyers stabilities involving a general control function, sum of powers of norms, product of powers of norms and mixed product-sum of powers of norms of the duodecic functional equation in quasi-β-normed spaces via fixed point method. We also illustrate a counter-example for non-stability of the duodecic functional equation in singular case.

2010 Mathematics Subject Classification. **39B82**. 39B72
Keywords. Quasi-β-normed spaces, Duodecic mapping, (β, p)-Banach spaces, Generalized Ulam-Hyers stability.

1 Introduction

The stability problem related to the stability of group homomorphisms of functional equations was originally introduced by Ulam [32] in 1940. The famous Ulam stability problem was partially solved by Hyers [14] for a linear functional equation of Banach spaces. Subsequently, the result of Hyers was generalized by Aoki [1], Bourgin [2], T. M. Rassias [26], J. M. Rassias ([20], [21], [22], [27]) and Gavruta [13]. During the last three decades, several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles on stabilities of various functional equations, one can see ([4], [5], [10], [11], [13], [23], [24], [28], [29], [25], [34], [38], [39], [40]) and monographs ([6], [8], [15]) and references therein.

Several mathematicians have remarked interesting applications of the Hyers-Ulam-Rassias stability theory to various mathematical problems. Stability theory is applied in fixed point theory to find the expression of the asymptotic derivative of a nonlinear operator. The stability results can be applied in stochastic analysis [18], financial and actuarial mathematics, as well as in psychology and sociology.

In this paper, we find the general solution of duodecic functional equation

$$\begin{aligned}&f(x+6y)-12f(x+5y)+66f(x+4y)\\&\quad-220f(x+3y)+495f(x+2y)-792f(x+y)\\&\quad\quad+924f(x)-792f(x-y)+495f(x-2y)-220f(x-3y)\\&\quad\quad\quad+66f(x-4y)-12f(x-5y)+f(x-6y)=479001600f(y).\end{aligned} \tag{1.1}$$

Moreover, we obtain the generalized Ulam-Hyers stability of the duodecic functional equation in quasi-β-normed spaces using fixed point method. Since $f(x)=x^{12}$ is a solution of (1.1), we say

Tbilisi Mathematical Journal 10(4) (2017), pp. 83–101.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 24 February 2017.
Accepted for publication: 25 September 2017.

that it is a duodecic functional equation. Every solution of the duodecic functional equation is said to be a duodecic mapping.

2 Preliminaries

In this section, we revoke some basic concepts concerning m-additive symmetric mappings, generalized polynomial and quasi-β-normed spaces. For detailed properties of m-additive symmetric mappings, the reader can refer ([7], [30], [31], [36], [37]).

Let X and Y be real vector spaces. A function $A : X \to Y$ is said to be additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in X$. It is easy to see that $A(rx) = rA(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$ (the set of rational numbers).

Let $n \in \mathbb{N}$ (the set of natural numbers). A function $A : X^n \to Y$ is called n-additive if it is additive in each of its variables. A function A_n is called symmetric if $A_n(x_1, x_2, \ldots, x_n) = A_n\left(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}\right)$ for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$ of $\{1, 2, \ldots, n\}$. If $A_n(x_1, x_2, \ldots, x_n)$ is n-additive symmetric map, then $A^n(x)$ will denote teh diagonal $A_n(x, x, \ldots, x)$ for $x \in X$ and note that $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$. Such a function $A^(x)$ will be called a monomial function of degree n (assuming $A^n \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_2 = \cdots = x_l = x$ and $x_{l+1} = x_{l+2} = \cdots = x_n = y$ in $A_n(x_1, x_2, \ldots, x_n)$ will be denoted by $A^{l,n-l}(x, y)$.

A function $p : X \to Y$ is called a generalized polynomial function of degree $n \in \mathbb{N}$ provided that there exist $A^0(x) = A^0 \in Y$ and i-additive symmetric functions $A_i : X^i \to Y$ (for $1 \le i \le n$) such that

$$p(x) = \sum_{i=0}^{n} A^i(x), \quad \text{for all } x \in X$$

and $A^n \not\equiv 0$.

For $f : X \to Y$, let Δ_h be the difference operator defined as follows:

$$\Delta_h f(x) = f(x + h) - f(x)$$

for $h \in X$. Furthermore, let $\Delta_h^0 f(x) = f(x)$, $\Delta_h^1 = \Delta_h f(x)$ and $\Delta_h \circ \Delta_h^n f(x) = \Delta_h^{n+1} f(x)$ for all $n \in \mathbb{N}$ and all $h \in X$. Here $\Delta_h \circ \Delta_h^n$ denotes the composition of the operators Δ_h and Δ_h^n. For any given $n \in \mathbb{N}$, the functional equation $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in X$ is well studied. In explicit form the last functional equation can be written as

$$\Delta_h^{n+1} f(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x + jh) = 0.$$

The following theorem was proved by Mazur and Orlicz, and in greater generality by Djoković (see [9]).

Theorem 2.1. Let X and Y be real vector spaces, $n \in \mathbb{N}$ and $f : X \to Y$, then the following are equivalent.

(1) $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in X$.

(2) $\Delta_{x_1,\ldots,x_{n+1}} f(x_0) = 0$ for all $x_0, x_1, \ldots, x_{n+1} \in X$.

(3) $f(x) = A^n(x) + A^{n-1}(x) + \cdots + A^2(x) + A^1(x) + A^0(x)$ for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)(i = 1, 2, \ldots, n)$ is the diagonal of an i-additive symmetric function $A_i : X^i \to Y$.

Let β be a fixed real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.2. Let X be a linear space over K. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following conditions:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that

$$\|x + y\| \leq K\left(\|x\| + \|y\|\right) \text{ for all } x, y \in X.$$

The pair $(X, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.3. A quasi-β-Banach space is a complete quasi-β-normed space.

Definition 2.4. A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p < 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

3 General solution of functional equation (1.1)

In this section, let X and Y be finite dimensional vector spaces. In the following theorem, we find the general solution of the duodecic functional equation (1.1).

Theorem 3.1. A function $f : X \to Y$ is a solution of the functional equation (1.1) if and only if f is of the form $f(x) = A^{12}(x)$ for all $x \in X$, where $A^{12}(x)$ is the diagonal of the 12-additive symmetric map $A_{12} : X^{12} \to Y$.

Proof. Assume that f satisfies the functional equation (1.1). Putting $x = y = 0$ in (1.1), we get $f(0) = 0$. Substituting $(x, y) = (x, x)$ and $(x, y) = (x, -x)$ in (1.1), and then subtracting the resulting equations, one finds

$$f(-x) = f(x) \tag{3.1}$$

for all $x \in X$. Plugging (x, y) into $(0, 2x)$ in (1.1) and then using (3.1) in the resulting equation, one obtains

$$f(12x) - 12f(10x) + 66f(8x) - 220f(6x) + 495f(4x) - 239501592f(2x) = 0 \tag{3.2}$$

for all $x \in X$. Letting (x, y) to $(6x, x)$ in (1.1) and then subtracting the resulting equation from (3.2), we have

$$\begin{aligned} 12f(11x) &- 78f(10x) + 220f(9x) - 429f(8x) + 792f(7x) \\ &- 1144f(6x) + 792f(5x) + 220f(3x) - 239501658f(2x) \\ &+ 479001612f(x) = 0 \end{aligned} \tag{3.3}$$

for all $x \in X$. Considering (x, y) as $(5x, x)$ in (1.1), multiplying by 12 and then subtracting the resulting equation from (3.3), we get

$$\begin{aligned} 66f(10x) &- 572f(9x) + 2211f(8x) - 5148f(7x) + 8360f(6x) - 10296f(5x) \\ &+ 9504f(4x) - 5720f(3x) - 239499018f(2x) + 6227020008f(x) = 0 \end{aligned} \tag{3.4}$$

for all $x \in X$. Taking (x, y) as $(4x, x)$ in (1.1), multiplying by 66 and then subtrcting the resulting equation from (3.4), one obtains

$$\begin{aligned} 220f(9x) &- 2145f(8x) + 9372f(7x) - 24310f(6x) + 41976f(5x) \\ &- 51480f(4x) + 46552f(3x) - 239531754f(2x) + 37841140920f(x) = 0 \end{aligned} \tag{3.5}$$

for all $x \in X$. Replacing (x, y) by $(3x, x)$ in (1.1), multiplying by 220 and then subtracting the resulting equation from (3.5), one finds

$$\begin{aligned} 495f(8x) &- 5148f(7x) + 24090f(6x) - 66924f(5x) + 122760f(4x) \\ &- 156948f(3x) - 239354874f(2x) + 143221380500f(x) = 0 \end{aligned} \tag{3.6}$$

for all $x \in X$. Substituting $(x, y) = (2x, x)$, multiplying by 495 and then subtracting the resulting equation from (3.6), we get

$$\begin{aligned} 792f(7x) &- 8580f(6x) + 41976f(5x) - 122760f(4x) \\ &+ 241032f(3x) - 239844924f(2x) + 380327673440f(x) = 0 \end{aligned} \tag{3.7}$$

for all $x \in X$. Plugging (x, y) into (x, x) in (1.1), multiplying by 792 and then subtracting the resulting equation from (3.7), we obtain

$$\begin{aligned} 924f(6x) - 11088f(5x) + 60984f(4x) - 203280f(3x) \\ - 239043420f(2x) + 759695816792f(x) = 0 \end{aligned} \tag{3.8}$$

for all $x \in X$. Letting (x, y) as $(0, x)$ in (1.1) and multiplying by 924, we find

$$\begin{aligned} 924f(6x) - 11088f(5x) + 60984f(4x) - 203280f(3x) \\ + 457380f(2x) - 221299471008f(x) = 0. \end{aligned} \tag{3.9}$$

for all $x \in X$. Subtracting (3.8) from (3.9), we arrive at

$$f(2x) = 4096f(x) = 2^{12}f(x) \tag{3.10}$$

for all $x \in X$.

On the other hand, one can rewrite the functional equation (1.1) in the form

$$\begin{aligned} f(x) + \frac{1}{924}f(x+6y) - \frac{1}{77}f(x+5y) + \frac{1}{14}f(x+4y) - \frac{55}{231}f(x+3y) \\ + \frac{495}{924}f(x+2y) - \frac{198}{231}f(x+y) - \frac{198}{231}f(x-y) + \frac{495}{924}f(x-2y) - \frac{55}{231}f(x-3y) \\ + \frac{1}{14}f(x-4y) - \frac{1}{77}f(x-5y) + \frac{1}{924}f(x-6y) - \frac{1}{518400}f(y) = 0, \end{aligned} \tag{3.11}$$

for all $x \in X$. By Theorems 3.5 and 3.6 in [37], f is a generalized polynomial function of degree at most 12, that is, f is of the form

$$\begin{aligned} f(x) = A^{12}(x) + A^{11}(x) + A^{10}(x) + A^{9}(x) + A^{8}(x) + A^{7}(x) + A^{6}(x) \\ + A^{5}(x) + A^{4}(x) + A^{3}(x) + A^{2}(x) + A^{1}(x) + A^{0}(x), \quad \forall x \in X, \end{aligned} \tag{3.12}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3, \ldots, 12$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and the function f is even. Thus we have $A^{11}(x) = A^9(x) = A^7(x) = A^5(x) = A^3(x) = A^1(x) = 0$. It follows that $f(x) = A^{12}(x) + A^{10}(x) + A^8(x) + A^6(x) + A^4(x) + A^2(x)$. By (3.10) and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain

$$\begin{aligned} 2^{12}\left(A^{12}(x) + A^{10}(x) + A^{8}(x) + A^{6}(x) + A^{4}(x) + A^{2}(x)\right) \\ = 2^{12}A^{12}(x) + 2^{10}A^{10}(x) + 2^{8}A^{8}(x) + 2^{6}A^{6}(x) + 2^{4}A^{4}(x) + 2^{2}A^{2}(x). \end{aligned}$$

Moreover, $2^{12}A^{10}(x) + 2^{12}A^{8}(x) = 2^{10}A^{10}(x) + 2^{8}A^{8}(x)$. Hence $A^{8}(x) = -\frac{4}{5}A^{10}(x)$. Also, $2^{10}A^{8}(x) + 2^{10}A^{6}(x) = 2^{8}A^{8}(x) + 2^{6}A^{6}(x)$ gives $A^{6}(x) = -\frac{4}{5}A^{8}(x) = \frac{16}{25}A^{10}(x)$. Similarly, $2^{8}A^{6}(x) + 2^{8}A^{4}(x) = 2^{6}A^{6}(x) + 2^{4}A^{4}(x)$ implies $A^{4}(x) = -\frac{4}{5}A^{6}(x) = -\frac{64}{125}A^{10}(x)$. Further, $2^{6}A^{4}(x) + 2^{6}A^{2}(x) = 2^{4}A^{4}(x) + 2^{2}A^{2}(x)$ shows $A^{2}(x) = -\frac{4}{5}A^{4}(x) = \frac{256}{625}A^{10}(x)$, for all $x \in X$. It follows that $A^{11}(x) = A^{9}(x) = A^{7}(x) = A^{5}(x) = A^{3}(x) = A^{1}(x) = 0$, $x \in X$. Therefore, $f(x) = A^{12}(x)$.

Conversely, assume that $f(x) = A^{12}(x)$ for all $x \in X$, where $A^{12}(x)$ is the diagonal of the 12-additive symmetric map $A_{12} : X^{12} \to Y$. From $A^{12}(x+y) = A^{12}(x) + A^{12}(y) + 12A^{11,1}(x,y) + 66A^{10,2}(x,y) + 220A^{9,3}(x,y) + 495A^{8,4}(x,y) + 792A^{7,5}(x,y) + 924A^{6,6}(x,y) + 792A^{5,7}(x,y) + 495A^{4,8}(x,y) + 220A^{3,9}(x,y) + 66A^{2,10}(x,y) + 12A^{1,11}A(x,y)$, $A^{12}(rx) = r^{12}A^{12}(x)$, $A^{11,1}(x,ry) = rA^{11,1}(x,y)$, $A^{10,2}(x,ry) = r^{2}A^{10,2}(x,y)$, $A^{9,3}(x,ry) = r^{3}A^{9,3}(x,y)$, $A^{8,4}(x,ry) = r^{4}A^{8,4}(x,y)$, $A^{7,5}(x,ry) = r^{5}A^{7,5}(x,y)$, $A^{6,6}(x,ry) = r^{6}A^{6,6}(x,y)$, $A^{5,7}(x,ry) = r^{7}A^{5,7}(x,y)$, $A^{4,8}(x,ry)$ $r^{8}A^{4,8}(x,y)$, $A^{3,9}(x,ry) = r^{9}A^{3,9}(x,y)$, $A^{2,10}(x,ry) = r^{10}A^{2,10}(x,y)$, $A^{1,11}(x,ry) = r^{11}A^{1,11}(x,y)$ $(x,y \in X, r \in \mathbb{Q})$, we see that f satisfies (1.1), which completes the proof of Theorem 3.1. Q.E.D.

4 Generalized Hyers-Ulam Stability of equation (1.1)

Throughout this section, we assume that X is a linear space and Y is a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$. For notational convenience, we define the difference operator for a given mapping $f : X \to Y$ as

$$\begin{aligned} D_t f(x,y) = {} & f(x+6y) - 12f(x+5y) + 66f(x+4y) - 220f(x+3y) + 495f(x+2y) \\ & - 792f(x+y) + 924f(x) - 792f(x-y) + 495f(x-2y) - 220f(x-3y) \\ & + 66f(x-4y) - 12f(x-5y) + f(x-6y) - 479001600f(y) \end{aligned}$$

for all $x, y \in X$.

Lemma 4.1. (see [34]). Let $i \in \{-1,1\}$ be fixed, $s, a \in \mathbb{N}$ with $a \geq 2$ and $\Psi : X \to [0,\infty)$ be a function such that there exists an $L < 1$ with $\Psi\left(a^i x\right) \leq a^{is\beta} L \Psi(x)$ for all $x \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|f(ax) - a^s f(x)\|_Y \leq \Psi(x) \tag{4.1}$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \to Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}\left|1 - L^i\right|}\Psi(x) \tag{4.2}$$

for all $x \in X$.

Theorem 4.2. Let $i \in \{-1,1\}$ be fixed. Let $\varphi : X \times X \to [0,\infty)$ be a function such that there exists an $L < 1$ with $\varphi\left(2^i x, 2^i y\right) \leq 4096^{i\beta} L\varphi(x,y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|D_t f(x,y)\|_Y \leq \varphi(x,y) \tag{4.3}$$

for all $x, y \in X$. Then there exists a unique duodecic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \leq \frac{1}{4096^\beta \left|1 - L^i\right|}\Psi(x) \tag{4.4}$$

for all $x \in X$, where

$$\begin{aligned}
&\Psi(x)\\
&= \frac{1}{239500800^\beta}\Big\{K^8\varphi(6x,x) + 12^\beta K^7\varphi(5x,x) + 66^\beta K^6\varphi(4x,x) + 220^\beta K^5\varphi(3x,x)\\
&+ 495^\beta K^4\varphi(2x,x) + 792^\beta K^3\varphi(x,x) + 924^\beta K^2\varphi(0,x) + \frac{K^8}{2^\beta\varphi(0,2x)}\\
&+ \Big(\frac{K^8}{1036800^\beta} + \frac{K^7}{479001600^\beta} + \frac{66^\beta K^6}{7257600^\beta} + \frac{220^\beta K^5}{2177280^\beta} + \frac{495^\beta K^4}{967680^\beta} + \frac{792^\beta K^3}{604800^\beta}\\
&+ \frac{924^\beta K^2}{1036800^\beta}\Big)\varphi(0,0) + \Big(\frac{12^\beta K^8}{479001600^\beta} + \frac{66^\beta K^7}{39916800^\beta} + \frac{220^\beta K^6}{7257600^\beta} + \frac{495^\beta K^5}{2177280^\beta}\\
&+ \frac{792^\beta K^4}{967680^\beta} + \frac{924^\beta K^3}{1209600^\beta}\Big)[\varphi(x,x) + \varphi(x,-x)] + \Big(\frac{K^{10}}{1209600^\beta} + \frac{66^\beta K^8}{479001600^\beta}\\
&+ \frac{220^\beta K^6}{39916800^\beta} + \frac{495^\beta K^6}{7257600^\beta} + \frac{792^\beta K^5}{2177280^\beta} + \frac{924^\beta K^4}{1935360^\beta}\Big)[\varphi(2x,2x) + \varphi(2x,-2x)]\\
&+ \Big(\frac{220^\beta K^8}{479001600^\beta} + \frac{495^\beta K^7}{39916800^\beta} + \frac{792^\beta K^6}{7257600^\beta} + \frac{924^\beta K^5}{4354560^\beta}\Big)[\varphi(3x,3x) + \varphi(3x,-3x)]\\
&+ \Big(\frac{K^{12}}{1935360^\beta} + \frac{495^\beta K^8}{479001600^\beta} + \frac{792^\beta K^7}{39916800^\beta} + \frac{924^\beta K^6}{14515200^\beta}\Big)[\varphi(4x,4x) + \varphi(4x,-4x)]\\
&+ \Big(\frac{792^\beta K^8}{479001600^\beta} + \frac{924^\beta K^7}{79833600^\beta}\Big)[\varphi(5x,5x) + \varphi(5x,-5x)]\\
&+ \Big(\frac{K^{12}}{4354560^\beta} + \frac{924^\beta K^9}{479001600^\beta}\Big)[\varphi(6x,6x) + \varphi(6x,-6x)]\\
&+ \frac{K^{13}}{14515200^\beta}[\varphi(8x,8x) + \varphi(8x,-8x)] + \frac{K^{14}}{79833600^\beta}[\varphi(10x,10x) + \varphi(10x,-10x)]\Big\}.
\end{aligned}$$

Proof. Substituting $x = y = 0$ in (4.3), one finds

$$\|f(0)\|_Y \leq \frac{1}{479001600^\beta}\varphi(0,0). \tag{4.5}$$

Replacing (x,y) by (x,x) and $(x,-x)$ in (4.3) and then subtracting the resulting equations, we obtain

$$\|f(x) - f(-x)\|_Y \leq \frac{K}{479001600^\beta}[\varphi(x,x) + \varphi(x,-x)] \tag{4.6}$$

for all $x \in X$. Plugging (x, y) into $(0, 2x)$ in (4.3) and using (4.5), (4.6) in the resulting equation, one gets

$$\begin{aligned}
&\|f(12x) - 12f(10x) + 66f(8x) - 220f(6x) + 495f(4x) - 239501592f(2x)\|_Y \\
&\quad \leq \frac{K}{2^\beta}\varphi(0, 2x) + \frac{K}{1036800^\beta}\varphi(0, 0) + \frac{K^3}{1209600^\beta}[\varphi(2x, 2x) + \varphi(2x, -2x)] \\
&\quad + \frac{K^4}{1935360^\beta}[\varphi(4x, 4x) + \varphi(4x, -4x)] + \frac{K^5}{4354560^\beta}[\varphi(6x, 6x) + \varphi(6x, -6x)] \\
&\quad + \frac{K^6}{14515200^\beta}[\varphi(8x, 8x) + \varphi(8x, -8x)] \\
&\qquad\qquad + \frac{K^7}{79833600^\beta}[\varphi(10x, 10x) + \varphi(10x, -10x)]
\end{aligned} \tag{4.7}$$

for all $x \in X$. Letting (x, y) as $(6x, x)$ in (4.3) and then subtracting the resulting equation from (4.7), we have

$$\begin{aligned}
&\Big\|12f(11x) - 78f(10x) + 220f(9x) - 429f(8x) + 792f(7x) - 1144f(6x) + 792f(5x) \\
&\qquad + 220f(3x) - 239501658f(2x) + 479001612f(x)\Big\|_Y \\
&\quad \leq K^2\varphi(6x, x) + \frac{K^2}{2^\beta}\varphi(0, 2x) + \left(\frac{K^2}{479001600^\beta} + \frac{K^2}{1036800^\beta}\right)\varphi(0, 0) \\
&\quad + \frac{K^4}{1209600^\beta}[\varphi(2x, 2x) + \varphi(2x, -2x)] + \frac{K^5}{1935360^\beta}[\varphi(4x, 4x) + \varphi(4x, -4x)] \\
&\quad + \frac{K^6}{4354560^\beta}[\varphi(6x, 6x) + \varphi(6x, -6x)] + \frac{K^7}{14515200^\beta}[\varphi(8x, 8x) + \varphi(8x, -8x)] \\
&\qquad\qquad + \frac{K^8}{79833600^\beta}[\varphi(10x, 10x) + \varphi(10x, -10x)]
\end{aligned} \tag{4.8}$$

for all $x \in X$. Replacing (x, y) by $(5x, x)$ in (4.3), multiplying by 12^β and then subtracting from (4.8), one obtains

$$\begin{aligned}
&\Big\|66f(10x) - 572f(9x) + 2211f(8x) - 5148f(7x) + 8360f(6x) - 10296f(5x) \\
&\qquad + 9504f(4x) - 5720f(3x) - 239499018f(2x) + 6227020008f(x)\Big\|_Y \\
&\quad \leq 12^\beta K^2\varphi(5x, x) + \frac{K^3}{2^\beta}\varphi(0, 2x)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{12^\beta K^3}{479001600^\beta}[\varphi(x,x)+\varphi(x,-x)] + \frac{K^5}{1209600^\beta}[\varphi(2x,2x)+\varphi(2x,-2x)]\\
&+ \frac{K^6}{1935360^\beta}[\varphi(4x,4x)+\varphi(4x,-4x)] + \frac{K^7}{4354560^\beta}[\varphi(6x,6x)+\varphi(6x,-6x)]\\
&+ \frac{K^8}{14515200^\beta}[\varphi(8x,8x)+\varphi(8x,-8x)]\\
&+ \frac{K^9}{79833600^\beta}[\varphi(10x,10x)+\varphi(10x,-10x)] \qquad (4.9)
\end{aligned}
$$

for all $x \in X$. Taking (x,y) as $(4x,x)$ in (4.3), multiplying by 66^β and then subtracting from (4.9), one finds

$$
\begin{aligned}
&\Big\| 220f(9x) - 2145f(8x) + 9372f(7x) - 24310f(6x) + 41976f(5x) - 51480f(4x)\\
&\quad + 46552f(3x) - 239531754f(2x) + 37841140920f(x)\Big\|_Y\\
&\le K^4\varphi(4x,x) + 12^\beta K^3\varphi(5x,x) + 66^\beta K^2\varphi(4x,x) + \frac{K^4}{2^\beta}\varphi(0,2x)\\
&+ \left(\frac{K^4}{1036800^\beta} + \frac{K^4}{479001600^\beta} + \frac{12^\beta K^3}{39916800^\beta} + \frac{66^\beta K^2}{7257600^\beta}\right)\varphi(0,0)\\
&+ \left(\frac{12^\beta K^4}{479001600^\beta} + \frac{66^\beta K^3}{39916800^\beta}\right)[\varphi(x,x)+\varphi(x,-x)]\\
&+ \left(\frac{K^6}{1209600^\beta} + \frac{66^\beta K^4}{479001600^\beta}\right)[\varphi(2x,2x)+\varphi(2x,-2x)]\\
&+ \frac{K^7}{1935360^\beta}[\varphi(4x,4x)+\varphi(4x,-4x)] + \frac{K^8}{4354560^\beta}[\varphi(6x,6x)+\varphi(6x,-6x)]\\
&+ \frac{K^9}{14515200^\beta}[\varphi(8x,8x)+\varphi(8x,-8x)]\\
&\quad + \frac{K^{10}}{79833600^\beta}[\varphi(10x,10x)+\varphi(10x,-10x)] \qquad (4.10)
\end{aligned}
$$

for all $x \in X$. Considering (x,y) as $(3x,x)$ in (4.3), multiplying by 220^β and then subtracting from

(4.10), we have

$$\begin{aligned}
&\Big\| 495f(8x) - 5148f(7x) + 24090f(6x) - 66924f(5x) + 122760f(4x) - 156948f(3x) \\
&\qquad - 239354874f(2x) + 143221369500f(x)\Big\|_Y \\
&\le K^5\varphi(6x,x) + 12^\beta K^4\varphi(5x,x) + 66^\beta K^3\varphi(4x,x) + 220^\beta K^2\varphi(3x,x) + \frac{K^5}{2^\beta}\varphi(0,2x) \\
&+ \left(\frac{K^5}{1036800^\beta} + \frac{K^5}{479001600^\beta} + \frac{12^\beta K^4}{39916800^\beta} + \frac{66^\beta K^3}{7257600^\beta} + \frac{220^\beta K^2}{2177280^\beta}\right)\varphi(0,0) \\
&+ \left(\frac{12^\beta K^5}{479001600^\beta} + \frac{66^\beta K^4}{39916800^\beta} + \frac{220^\beta K^3}{7257600^\beta}\right)[\varphi(x,x) + \varphi(x,-x)] \\
&+ \left(\frac{K^7}{1209600^\beta} + \frac{66^\beta K^5}{479001600^\beta} + \frac{220^\beta K^4}{39916800^\beta}\right)[\varphi(2x,2x) + \varphi(2x,-2x)] \\
&+ \frac{220^\beta K^5}{479001600^\beta}[\varphi(3x,3x) + \varphi(3x,-3x)] + \frac{K^8}{1935360^\beta}[\varphi(4x,4x) + \varphi(4x,-4x)] \\
&+ \frac{K^9}{4354560^\beta}[\varphi(6x,6x) + \varphi(6x,-6x)] + \frac{K^{10}}{14515200^\beta}[\varphi(8x,8x) + \varphi(8x,-8x)] \\
&+ \frac{K^{11}}{79833600^\beta}[\varphi(10x,10x) + \varphi(10x,-10x)]
\end{aligned} \tag{4.11}$$

for all $x \in X$. Replacing (x, y) by $(2x, x)$ in (4.3), multiplying by 495^β and then subtracting from (4.11), we get

$$\begin{aligned}
&\Big\| 792f(7x) - 8580f(6x) + 41976f(5x) - 122760f(4x) + 241032f(3x) \\
&\qquad - 239844924f(2x) + 380327662440f(x)\Big\|_Y \\
&\le K^6\varphi(6x,x) + 12^\beta K^5\varphi(5x,x) + 66^\beta K^4\varphi(4x,x) + 220^\beta K^3\varphi(3x,x) \\
&+ 495^\beta K^2\varphi(2x,x) + \frac{K^6}{2^\beta}\varphi(0,2x) \\
&+ \Big(\frac{K^6}{1036800^\beta} + \frac{K^6}{479001600^\beta} + \frac{12^\beta K^5}{39916800^\beta} \\
&\qquad + \frac{66^\beta K^4}{7257600^\beta} + \frac{220^\beta K^3}{2177280^\beta} + \frac{495^\beta K^2}{967680^\beta}\Big)\varphi(0,0)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{12^\beta K^6}{479001600^\beta}+\frac{66^\beta K^5}{39916800^\beta}\\
&\qquad+\frac{220^\beta K^4}{7257600^\beta}+\frac{495^\beta K^3}{2177280^\beta}\Big)[\varphi(x,x)+\varphi(x,-x)]\\
&+\Big(\frac{K^8}{1209600^\beta}+\frac{66^\beta K^6}{479001600^\beta}+\frac{220^\beta K^5}{39916800^\beta}+\frac{495^\beta K^3}{2177280^\beta}\Big)[\varphi(2x,2x)+\varphi(2x,-2x)]\\
&+\left(\frac{220^\beta K^6}{479001600^\beta}+\frac{495^\beta K^5}{39916800^\beta}\right)[\varphi(3x,3x)+\varphi(3x,-3x)]\\
&+\left(\frac{K^9}{193360^\beta}+\frac{495^\beta K^6}{479001600^\beta}\right)[\varphi(4x,4x)+\varphi(4x,-4x)]\\
&+\frac{K^{10}}{4354560^\beta}[\varphi(6x,6x)+\varphi(6x,-6x)]+\frac{K^{11}}{14515200^\beta}[\varphi(8x,8x)+\varphi(8x,-8x)]\\
&\qquad+\frac{K^{12}}{79833600^\beta}[\varphi(10x,10x)+\varphi(10x,-10x)]
\end{aligned}\tag{4.12}
$$

for all $x\in X$. Substituting (x,y) as (x,x) in (4.3), multiplying by 792^β and then subtracting from (4.12), we obtain

$$
\begin{aligned}
&\Big\|924f(6x)-11088f(5x)-60984f(4x)-203280f(3x)-239043420f(2x)\\
&\qquad+759695805792f(x)\Big\|_Y\\
&\le K^7\varphi(6x,x)+12^\beta K^6\varphi(5x,x)+66^\beta K^5\varphi(4x,x)+220^\beta K^4\varphi(3x,x)\\
&+495^\beta K^3\varphi(2x,x)+792^\beta K^2\varphi(x,x)+\frac{K^7}{2^\beta}\varphi(0,2x)\\
&+\Big(\frac{K^7}{1036800^\beta}+\frac{K^6}{479001600^\beta}+\frac{12^\beta K^6}{39916800^\beta}+\frac{66^\beta K^5}{7257600^\beta}+\frac{220^\beta K^4}{2177280^\beta}+\frac{495^\beta K^3}{967680^\beta}\\
&\quad+\frac{792^\beta K^2}{604800^\beta}\Big)\varphi(0,0)+\Big(\frac{12^\beta K^7}{479001600^\beta}+\frac{66^\beta K^6}{39916800^\beta}+\frac{220^\beta K^5}{7257600^\beta}\\
&\qquad+\frac{495^\beta K^4}{2177280^\beta}+\frac{792^\beta K^3}{967680^\beta}\Big)[\varphi(x,x)+\varphi(x,-x)]\quad+\Big(\frac{K^9}{1209600^\beta}+\frac{66^\beta K^7}{479001600^\beta}\\
&+\frac{220^\beta K^5}{39916800^\beta}+\frac{495^\beta K^5}{7257600^\beta}+\frac{7926\beta K^4}{2177280^\beta}\Big)[\varphi(2x,2x)+\varphi(2x,-2x)]\\
&+\left(\frac{220^\beta K^7}{479001600^\beta}+\frac{495^\beta K^6}{39916800^\beta}+\frac{792^\beta K^5}{7257600^\beta}\right)[\varphi(3x,3x)+\varphi(3x,-3x)]\\
&+\left(\frac{K^{10}}{1935360^\beta}+\frac{495^\beta K^7}{479001600^\beta}+\frac{792^\beta K^6}{39916800^\beta}\right)[\varphi(4x,4x)+\varphi(4x,-4x)]\\
&+\frac{792^\beta K^7}{479001600^\beta}[\varphi(5x,5x)+\varphi(5x-5x)]+\frac{K^{11}}{4354560^\beta}[\varphi(6x,6x)+\varphi(6x,-6x)]
\end{aligned}
$$

$$+\frac{K^{12}}{14515200^{\beta}}[\varphi(8x,8x)+\varphi(8x,-8x)] \\ +\frac{K^{13}}{79833600^{\beta}}[\varphi(10x,10x)+\varphi(10x,-10x)] \tag{4.13}$$

for all $x \in X$. Plugging (x,y) into $(0,x)$ in (4.3), multiplying by 924^{β} and then subtracting from (4.13), we arrive at

$$\left\|f(x)-2^{12}f(x)\right\|_Y \leq \Psi(x)$$

for all $x \in X$. By Lemma 4.1, there exists a unique mapping $T : X \to Y$ such that $T(2x) = 2^{12}T(x)$ and

$$\|f(x)-T(x)\|_Y \leq \frac{1}{4096^{\beta}\left|1-L^i\right|}\Psi(x)$$

for all $x \in X$. It remains to show that T is a duodecic map. By (4.3), we have

$$\begin{aligned}\left\|\frac{1}{4096^{in}}D_t f\left(2^{in}x,2^{in}y\right)\right\|_Y &\leq 4096^{-in\beta}\varphi\left(2^{in}x,2^{in}y\right) \\ &\leq 4096^{-in\beta}\left(4096^{i\beta}L\right)^n\varphi(x,y) \\ &= L^n\varphi(x,y)\end{aligned}$$

for all $x,y \in X$ and $n \in \mathbb{N}$. So $\|D_tT(x,y)\|_Y = 0$ for all $x,y \in X$. Thus the mapping $T : X \to Y$ is duodecic. Q.E.D.

Corollary 4.3. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let c_1, a be positive numbers with $a \neq \frac{12\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\|D_tf(x,y)\|_Y \leq c_1\left(\|x\|_X^a+\|y\|_X^a\right)$$

for all $x,y \in X$. Then there exists a unique duodecic mapping $T : X \to Y$such that

$$\|f(x)-T(x)\|_Y \leq \begin{cases}\frac{c_1\delta_a}{4096^{\beta}-2^{a\alpha}}\|x\|_X^a, & a \in \left(0,\frac{12\beta}{\alpha}\right)\\ \frac{2^{a\alpha}c_1\delta_a}{4096^{\beta}(2^{a\alpha}-4096^{\beta})}\|x\|_X^a, & a \in \left(\frac{12\beta}{\alpha},\infty\right)\end{cases}$$

for all $x \in X$, where

$$\delta_a = \frac{1}{239500800^\beta}\Big\{K^8\left(6^{a\alpha}+1\right)+12^\beta K^7\left(5^{a\alpha}+1\right)+66^\beta K^6\left(4^{a\alpha}+1\right)$$
$$+495^\beta K^4\left(2^{a\alpha}+1\right)+2\cdot 792^\beta K^3+924^\beta K^2+\frac{K^8 2^{a\alpha}}{2^\beta}$$
$$+2\cdot 2^{a\alpha}\Big(\frac{12^\beta K^8}{479001600^\beta}+\frac{66^\beta K^7}{39916800^\beta}+\frac{220^\beta K^6}{7257600^\beta}+\frac{495^\beta K^5}{2177280^\beta}$$
$$+\frac{792^\beta K^4}{967680^\beta}\frac{924^\beta K^3}{1209600^\beta}\Big)+4\cdot 2^{a\alpha}\Big(\frac{K^{10}}{1209600^\beta}+\frac{66^\beta K^8}{479001600^\beta}+\frac{220^\beta K^6}{39916800^\beta}$$
$$+\frac{495^\beta K^6}{7257600^\beta}+\frac{792^\beta K^5}{2177280^\beta}+\frac{924^\beta K^4}{1935360^\beta}\Big)$$
$$+4\cdot 3^{a\alpha}\left(\frac{220^\beta K^8}{479001600^\beta}+\frac{495^\beta K^7}{399916800^\beta}+\frac{792^\beta K^6}{7257600^\beta}+\frac{924^\beta K^5}{4354560^\beta}\right)$$
$$+4\cdot 4^{a\alpha}\left(\frac{K^{12}}{1935360^\beta}+\frac{495^\beta K^8}{479001600^\beta}+\frac{792^\beta K^7}{39916800^\beta}+\frac{924^\beta K^6}{14515200^\beta}\right)$$
$$+4\cdot 5^{a\alpha}\left(\frac{792^\beta K^8}{479001600^\beta}+\frac{924^\beta K^7}{79833600^\beta}\right)+4\cdot 6^{a\alpha}\left(\frac{K^{12}}{4354560^\beta}+\frac{924^\beta K^9}{479001600^\beta}\right)$$
$$+\frac{4\cdot K^{13}}{14515200^\beta}+\frac{4\cdot 10^{a\alpha}K^{14}}{79833600^\beta}\Big\}.$$

Proof. The proof is obtained by taking $\varphi(x,y) = c_1\left(\|x\|_X^a+\|y\|_X^a\right)$, for all $x, y \in X$ and $L = \frac{2^{\alpha\lambda}}{4096^\beta}$ in Theorem 4.2. Q.E.D.

Corollary 4.4. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let c_2, r, s be positive numbers with $a = r+s \neq \frac{12\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\|D_t f(x,y)\|_Y \leq c_2\,\|x\|_X^r\,\|y\|_X^s$$

for all $x, y \in X$. Then there exists a unique duodecic mapping $T : X \to Y$such that

$$\|f(x)-T(x)\|_Y \leq \begin{cases} \frac{c_2\delta_{r,s}}{4096^\beta-2^{a\alpha}}\,\|x\|_X^a\,, & a\in\left(0,\frac{12\beta}{\alpha}\right)\\ \frac{2^{a\alpha}c_2\delta_{r,s}}{4096^\beta(2^{a\alpha}-4096^\beta)}\,\|x\|_X^a\,, & a\in\left(\frac{12\beta}{\alpha},\infty\right)\end{cases}$$

for all $x \in X$, where

$$\delta_{r,s} = \frac{1}{239500800^\beta}\Big\{K^8 6^{r\alpha}+12^\beta K^7 5^{r\alpha}+66^\beta K^6 4^{r\alpha}+220^\beta K^\beta 3^{r\alpha}+495^\beta K^4 2^{r\alpha}$$
$$+792^\beta K^3+2\Big(\frac{12^\beta K^8}{479001600^\beta}+\frac{66^\beta K^7}{39916800^\beta}+\frac{220^\beta K^6}{7257600^\beta}+\frac{795^\beta K^5}{2177280^\beta}+\frac{792^\beta K^4}{967680^\beta}$$
$$+\frac{924^\beta K^3}{1209600^\beta}\Big)+2\cdot 2^{a\alpha}\Big(\frac{K^{10}}{1209600^\beta}+\frac{66^\beta K^8}{479001600^\beta}+\frac{220^\beta K^6}{39916800^\beta}+\frac{495^\beta K^6}{7257600^\beta}$$

$$+\frac{792^{\beta}K^{5}}{2177280^{\beta}}+\frac{924^{\beta}K^{4}}{1935360^{\beta}}\Big)+2\cdot 3^{a\alpha}\Big(\frac{220^{\beta}K^{8}}{479001600^{\beta}}+\frac{495^{\beta}K^{7}}{39916800^{\beta}}+\frac{792^{\beta}K^{6}}{7257600^{\beta}}$$
$$+\frac{924^{\beta}K^{5}}{4354560^{\beta}}\Big)+2\cdot 4^{a\alpha}\left(\frac{K^{12}}{1935360^{\beta}}+\frac{495^{\beta}K^{8}}{479001600^{\beta}}+\frac{79^{\beta}K^{7}}{39916800^{\beta}}+\frac{924^{\beta}K^{6}}{14515200^{\beta}}\right)$$
$$+2\cdot 5^{a\alpha}\left(\frac{792^{\beta}K^{8}}{479001600^{\beta}}+\frac{924^{\beta}K^{7}}{79833600^{\beta}}\right)+2\cdot 6^{a\alpha}\left(\frac{K^{12}}{4354560^{\beta}}+\frac{924^{\beta}K^{9}}{479001600^{\beta}}\right)$$
$$+\frac{2\cdot 8^{a\alpha}K^{13}}{14515200^{\beta}}+\frac{2\cdot 10^{a\alpha}K^{14}}{79833600^{\beta}}\Big\}.$$

Proof. Letting $\varphi(x,y)=c_2\left\|x\right\|_X^r\left\|y\right\|_X^s$, for all $x,y\in X$ and $L=\frac{2^{a\alpha}}{4096^{\beta}}$ in Theorem 4.2, we obtain the required results. Q.E.D.

Corollary 4.5. Let X be a quasi-α-normed space with quasi-α-norm $\left\|\cdot\right\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\left\|\cdot\right\|_Y$. Let c_3,r,s be positive numbers with $a=r+s\neq\frac{12\beta}{\alpha}$ and $f:X\to Y$ be a mapping satisfying

$$\left\|D_t f(x,y)\right\|_Y\leq c_3\left[\left\|x\right\|_X^r\left\|y\right\|_X^s+\left(\left\|x\right\|_X^{r+s}+\left\|y\right\|_X^{r+s}\right)\right]$$

for all $x,y\in X$. Then there exists a unique duodecic mapping $T:X\to Y$such that

$$\left\|f(x)-T(x)\right\|_Y\leq\begin{cases}\frac{c_3(\delta_{r,s}+\delta_a)}{4096^{\beta}-2^{a\alpha}}\left\|x\right\|_X^a, & a\in\left(0,\frac{12\beta}{\alpha}\right)\\ \frac{2^{a\alpha}c_3(\delta_{r,s}+\delta_a)}{4096^{\beta}(2^{a\alpha}-4096^{\beta})}\left\|x\right\|_X^a, & a\in\left(\frac{12\beta}{\alpha},\infty\right)\end{cases}$$

for all $x\in X$, where $\delta_{r,s}$ and δ_a are defined as in Corollaries in 4.4 and 4.3.

Proof. By taking $\varphi(x,y)=c_3\left[\left\|x\right\|_X^r\left\|y\right\|_X^s+\left(\left\|x\right\|_X^{r+s}+\left\|y\right\|_X^{r+s}\right)\right]$, for all $x,y\in X$ and $L=\frac{2^{\alpha\lambda}}{4096^{\beta}}$ in Theorem 4.2, we arrive at the desired results. Q.E.D.

5 Counter-example

In this section, using the idea of the well-known counter-example provided by Z. Gajda [12], we illustrate a counter-example that the functional equation (1.1) is not stable for $a=12$ in Corollary 4.3.

We consider the function

$$\varphi(x)=\begin{cases}x^{12}, & \text{for } |x|<1\\ 1, & \text{for } |x|\geq 1.\end{cases}\tag{5.1}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\sum_{n=0}^{\infty}2^{-12n}\varphi(2^n x)\tag{5.2}$$

for all $x\in\mathbb{R}$. The function f serves as a counter-example for the fact that the functional equation (1.1) is not stable for $a=12$ in Corollary 4.3 in the following theorem.

Theorem 5.1. If the function f defined in (5.2) satisfies the functional inequality

$$|D_u f(x,y)| \leq \frac{479005696 \cdot 4096^3}{4095}\left(|x|^{12}+|y|^{12}\right) \tag{5.3}$$

for all $x, y \in \mathbb{R}$, then there do not exist an duodecic mapping $T : \mathbb{R} \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|f(x) - T(x)| \leq \delta\, |x|^{12}, \qquad \text{for all } x \in \mathbb{R}.$$

Proof. First, we are going to show that f satisfies (5.3).

$$|f(x)| = \left|\sum_{n=0}^{\infty} 2^{-12n}\varphi(2^n x)\right| \leq \sum_{n=0}^{\infty}\frac{1}{2^{12n}} = \frac{4096}{4095}.$$

Therefore, we see that f is bounded by $\frac{4096}{4095}$ on $\mathbb{R}$. If $|x|^{12}+|y|^{12}=0$ or $|x|^{12}+|y|^{12} \geq \frac{1}{4096}$, then

$$|D_t f(x,y)| \leq \frac{(479005696)(4096)}{4095} \leq \frac{(479005696)(4096)^2}{4095}\left(|x|^{12}+|y|^{12}\right).$$

Now, suppose that $0 < |x|^{12}+|y|^{12} < \frac{1}{4096}$. Then there exists a non-negative integer k such that

$$\frac{1}{4096^{k+1}} \leq |x|^{12}+|y|^{12} < \frac{1}{4096^k}. \tag{5.4}$$

Hence $4096^k\,|x|^{12} < \frac{1}{4096}$, $4096^k\,|y|^{12} < \frac{1}{4096}$ and $2^n(x+6y)$, $2^n(x+5y)$, $2^n(x+4y)$, $2^n(x+3y)$, $2^n(x+2y)$, $2^n(x+y)$, $2^n(x)$, $2^n(y)$, $2^n(x-y)$, $2^n(x-2y)$, $2^n(x-3y)$, $2^n(x-4y)$, $2^n(x-5y)$, $2^n(x-6y) \in (-1,1)$ for all $n = 0,1,2,\ldots,k-1$. Hence for $n = 0,1,2,\ldots,k-1$,

$$\begin{aligned}
&\varphi\left(2^n(x+6y)\right) - 12\varphi\left(2^n(x+5y)\right) + 66\varphi\left(2^n(x+4y)\right) - 220\varphi\left(2^n(x+3y)\right)\\
&\quad + 495\varphi\left(2^n(x+2y)\right) - 792\varphi\left(2^n(x+y)\right) + 924\varphi\left(2^n x\right) - 792\varphi\left(2^n(x-y)\right)\\
&\quad + 495\varphi\left(2^n(x-2y)\right) - 220\varphi\left(2^n(x-3y)\right) + 66\varphi\left(2^n(x-4y)\right) - 12\varphi\left(2^n(x-5y)\right)\\
&\qquad + \varphi\left(2^n(x-6y)\right) - 39916800\varphi\left(2^n y\right) = 0.
\end{aligned} \tag{5.5}$$

From the definition of f and the inequality (5.4), we obtain that

$$\begin{aligned}
|D_t f(x,y)| = \Big| & \sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x+6y)\right) - 12\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x+5y)\right) \\
& + 66\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x+4y)\right) - 220\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x+3y)\right) \\
& + 495\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x+2y)\right) - 792\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x+y)\right) \\
& + 924\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n x\right) - 792\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x-y)\right) \\
& + 495\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x-2y)\right) - 220\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x-3y)\right) \\
& + 66\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x-4y)\right) - 12\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x-5y)\right) \\
& + \sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n(x-6y)\right) - 479001600\sum_{n=0}^{\infty} 2^{-12n}\varphi\left(2^n y\right) \Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=k}^{\infty} 2^{-12n} \cdot 479005696 \leq 479005695 \cdot \frac{2^{12(1-k)}}{4095} \\
&\leq \frac{479005696 \cdot 4096^3}{4095}\left(|x|^{12} + |y|^{12}\right). \qquad (5.6)
\end{aligned}$$

Therefore, f satisfies (5.3) for all $x, y \in \mathbb{R}$. Now, we claim that the functional equation (1.1) is not stable for $a = 12$ in Corollary 4.3 ($\alpha = \beta = p = 1$). Suppose on the contrary that there exists a duodecic mapping $T : \mathbb{R} \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|f(x) - T(x)| \leq \delta\,|x|^{12}, \quad \text{for all } x \in \mathbb{R}.$$

Then there exists a constant $c \in \mathbb{R}$ such that $T(x) = cx^{12}$ for all rational numbers x (see [17]). So we obtain that

$$|f(x)| \leq (\delta + |c|)\,|x|^{12} \qquad (5.7)$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m+1 > \delta+|c|$. If x is a rational number in $(0, 2^{-m})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \ldots, m$, and for this x, we get

$$f(x) = \sum_{n=0}^{\infty} 2^{-12n}\varphi(2^n x) \geq \sum_{n=0}^{m} 2^{-12n}\left(2^n x\right)^{12} = (m+1)x^{12} > (\delta + |c|)x^{12} \qquad (5.8)$$

which contradicts (5.7). Hence the functional equation (1.1) is not stable for $a = 12$ in Corollary 4.3. Q.E.D.

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